THE $k$-FOLD LIST COLORING OF CYCLES WITH HALL’S CONDITION

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Abstract

We prove that any cycle $C_n$, $n \geq 4$, with list assignment $L$, has a $k$-fold list coloring from the given lists if (i) each list contains at least $2k$ colors and (ii) $C_n$ and $L$ satisfy Hall’s condition for $k$-fold list colorings. Further, $2k$ in (i) cannot be replaced by $2k - 1$ if either $n$ is odd, or $n$ is even and $n \geq k + 2$. In other words, if $n \geq 4$, the $k$-fold Hall number of a cycle $C_n$ satisfies $h^{(k)}(C_n) \leq 2k$, with equality if $n$ is odd, or $n$ is even and $n \geq k + 2$.

Keywords: Hall’s condition, Hall number, list coloring, list assignment, $k$-fold choice number, $k$-fold chromatic number.

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1. Introduction

A list assignment, or a color supply, for a graph $G = (V, E)$ is an assignment to the vertices of $G$ of finite subsets (“lists”) of a set $C$ of colors. A color demand for $G$ is an assignment of positive integers to the vertices of $G$. If $L$ is a color supply and $w$ is a color demand for $G$, an $(L, w)$-coloring of $G$ is a function $\varphi$ which assigns to each $v \in V$ a subset $\varphi(v) \subseteq L(v)$, with $|\varphi(v)| = w(v)$, such that $\varphi(u) \cap \varphi(v) = \emptyset$ whenever $uv$ is an edge in $G$. When $w$ is a constant function, say $w = k$, we let the value $k$ stand for the function $w$ and we may refer to an $(L, k)$-coloring of $G$ as a $k$-fold list coloring from $L$.

For any positive integer $k$ the $k$-fold chromatic number of $G$, denoted $\chi^{(k)}(G)$, is the smallest integer $m$ such that there is a $k$-fold list coloring from the constant color supply $L \equiv \{1, \ldots, m\}$. The $k$-fold choice number of $G$, denoted $ch^{(k)}(G)$, is the smallest integer $m$ such that there is a $k$-fold list coloring from any color supply $L$ satisfying $|L(v)| \geq m$ for all $v \in V$.

The idea of coloring the vertices of a graph with subsets of a fixed set arguably originated with Hilton, Rado, Scott, or Stahl ([14],[17],[19]). The $k$-fold coloring of cycles is a particular case of edge coloring of multicycles which was studied by Kostochka and Woodall in [15]. It turns out that for cycles the $k$-fold chromatic and choice numbers are the same: Tuza and Voigt in [20] and Gutner and Tarsi in [6] proved that $ch^{(k)}(C_{2m}) = 2k$, $m = 1, 2, \ldots$, $k = 1, 2, \ldots$, and Slivnik [18] used measure theoretic methods to show that $ch^{(k)}(C_{2m+1}) = 2k + \left\lceil \frac{k}{m} \right\rceil$; these values had long been known for $\chi^{(k)}(G)$. Our aim here is to give an upper bound for cycles, with equality in many cases, on another $k$-fold list coloring parameter whose definition involves a fairly well-known necessary condition for the existence of an $(L, w)$-coloring of $G$ called Hall’s condition. Our main result will imply the results above about $ch^{(k)}$ as a corollary.

Given $G, L, w$, an induced subgraph $H$ of $G$, and a color $x \in C$, let $H(x, L)$ denote the subgraph of $H$ induced by $\{v \in V(H) | x \in L(v)\}$ and let $\alpha(H(x, L))$ denote the vertex independence number of this subgraph. [If $x$ does not appear on any lists $L(v)$, $v \in V(H)$, set $\alpha(H(x, L)) = 0$.] If there is an $(L, w)$-coloring $\varphi$ of $G$ then the set of vertices $v \in V(H)$ such that $x \in \varphi(v)$ is an independent set of vertices in $H(x, L)$. Therefore, there can be no more than $\alpha(H(x, L))$ appearances of $x$ in the color sets for vertices in $H$. Since the total number of appearances of all symbols in those color sets is

$$\sum_{x \in C} |\varphi(v)| = \sum_{v \in V(H)} w(v),$$

(*)

we have

$$\sum_{x \in C} \alpha(H(x, L)) \geq \sum_{v \in V(H)} w(v).$$

If (*) holds for every induced subgraph $H$ of $G$, then $G, L$ and $w$ satisfy Hall’s condition. (NOTE: If $G, L$ and $w$ satisfy Hall’s condition, then (*) holds for every subgraph $H$ of $G$, induced or not.) Observe that when $w = k$, a constant, the right hand side of (*) is $k|V(H)|$. 


Hall’s condition is so named because it was inspired by Hall’s theorem on systems of distinct representatives [7], which can be viewed as a list coloring theorem about the special case when \( G \) is a clique and \( w = 1 \) [10]. Further, the improvement of Hall’s theorem, in which the requirement that \( w = 1 \) is removed, due to Rado [16] and, independently, to Halmos and Vaughan [8], can be stated thus: when \( G \) is a clique, Hall’s condition suffices for the existence of an \((L, w)\)-coloring of \( G \).

For each positive integer \( k \), the \( k \)-fold Hall number of \( G \), denoted \( h^{(k)}(G) \), is the smallest integer \( m \geq k \) such that there is an \((L, w)\)-coloring of \( G \) whenever \( G, L \) and \( k \) satisfy Hall’s condition and \( |L(v)| \geq m \) for all \( v \in V \). Note that \( h^{(k)}(G) \geq k \) for every \( G \) and \( k \) and the Hall-Rado-Halmos-Vaughan theorem referred to above implies equality, for every \( k \), when \( G \) is a clique. It follows from results in [2], [5] and [3] that \( h^{(k)}(G) = k \) for \( k = 1, 2, \ldots \) if and only if \( G \) is the line graph of a forest.

Clearly \( h^{(k)}(G) \leq ch^{(k)}(G) \) for all \( k \) and \( G \), but \( h^{(k)}(G) < \chi^{(k)}(G) \), \( h^{(k)}(G) = \chi^{(k)}(G) \), and \( h^{(k)}(G) > \chi^{(k)}(G) \) are all possible [5]. In [5] it is shown that \( h^{(k)}(G) = ch^{(k)}(G) \) if either is larger than \( ch^{(k)}(G) \). This result may have its uses, since both \( ch^{(k)}(G) \) and \( h^{(k)}(G) \) are difficult to determine.

For \( k \geq 2 \), the problem of determining the \( k \)-fold Hall numbers of trees is solved in [3]:

If \( r \geq 3 \), \( h^{(k)}(K_{1, r}) = 2k - \left\lceil \frac{k}{r} \right\rceil \); if \( r \leq 3 \), \( h^{(k)}(K_{1, r}) = k \); and if \( T \) is a tree which is not a star, \( h^{(k)}(T) = 2k \). It follows from the main result in [10] that \( h^{(1)}(T) = 1 \) for all trees \( T \).

Regarding cycles, since \( C_3 = K_3 \) we have \( h^{(k)}(C_3) = k \) for all \( k = 1, 2, \ldots \). In [4], we proved that \( h^{(k)}(C_4) = \left\lceil \frac{5k}{3} \right\rceil \) for all \( k = 1, 2, \ldots \). From [13] we have \( h^{(1)}(C_n) = 2 \) for all \( n > 3 \). Our main result is the following:

**Theorem 1.1.** For any integers \( n \geq 4 \) and \( k \geq 1 \), \( h^{(k)}(C_n) \leq 2k \), with equality if \( n \) is odd, or if \( n \) is even and \( n \geq k + 2 \).

For \( n \) even the inequality \( h^{(k)}(C_n) \leq 2k \) follows from \( h^{(k)} \leq ch^{(k)} \) and the previously cited result from [6] and [20] that \( ch^{(k)}(C_n) = 2k \). But our proof will be independent of all direct proofs of this fact, and that allows us to turn the tables and give a new proof of both old results about \( ch^{(k)}(C_n) \), \( n \) even or odd. The following is a straightforward consequence of the previously mentioned result in [5] that \( ch^{(k)} = \max\{\chi^{(k)}, h^{(k)}\} \), of the well known values of \( \chi^{(k)}(C_n), k \geq 1, n \geq 3 \), and of the theorem.

**Corollary 1.2.** For any integers \( k \geq 1, m \geq 2 \), \( ch^{(k)}(C_{2m}) = 2k \) and \( ch^{(k)}(C_{2m+1}) = 2k + \left\lceil \frac{5k}{3} \right\rceil \).

In the case of \( C_{2m+1} \) the proof of the corollary constitutes a purely combinatorial proof of the result proved by Slivnik [18] with the involvement of Lebesgue measure.

There is an open problem in [5] that our results bear on: does \( \lim_{k \to \infty} \frac{h^{(k)}(G)}{k} \) exist for every graph \( G \)? To the two classes of graphs for which the answer was known to be
yes–line graphs of forests and trees–we can now add: odd cycles. It feels strange to wrestle with a list coloring problem that is apparently harder for even cycles than for odd cycles! We throw further light on $h^k(C_{2m})$ when $2m \leq k+1$ in [4].

2. Proof of the Theorem

The proof that $h^k(C_n) \leq 2k$ will use the following, a special case of the main result in [2], first proved in [1].

**Path Lemma.** For any path $P$ with color demand $w$ and color supply $L, P$ is $(L, w)$-colorable if and only if $P, L$ and $w$ satisfy Hall’s condition.

It may be worth mentioning that in [1] there is an efficient algorithm for either coloring $P$ or discovering that Hall’s condition is not satisfied.

Our strategy for proving that $h^k(C_n) \leq 2k$ will use the Path Lemma thus: we will show that when $C_n, L$ and $k$ satisfy Hall’s condition and $|L(v)| \geq 2k$ for all $v \in V(C_n)$, then for some $v \in V(C_n)$ there is a $k$-set $S \subseteq L(v)$ such that if $L'$ is defined on the path $C_n - v$ by removing all of the elements of $S$ from the lists on the neighbors of $v$, and otherwise putting $L' = L$, then $C_n - v$, $L'$ and $k$ satisfy Hall’s condition. Coloring $v$ with $S$ and putting this with an $(L', k)$-coloring of $C_n - v$ then produces an $(L, k)$-coloring of $C_n$.

It will be useful to note as in [1] that for any list assignment $L$ to $C_n$ and any $x \in C$, we may as well suppose that $C_n(x, L)$ is connected. The reason: for any graph $G$, color supply $L$, and color demand $w$, if $G(x, L)$ is disconnected for some $x \in C$, we can make a new lists assignment $\hat{L}$ by replacing $x$ on the lists on each component of $G(x, L)$ by a new symbol, so that $x$’s replacements on those components are different from each other and from all the other symbols appearing on lists on $G$. It is straightforward to see that $G, L$ and $w$ satisfy Hall’s condition if and only if $G, \hat{L}$ and $w$ do, and that there is an $(L, w)$-coloring of $G$ if and only if there is an $(\hat{L}, w)$-coloring of $G$.

Given a vertex $v \in V(G) = C_n$ and a color $x \in L(v)$, we say that $x$ is bad at $v$ if and only if no maximum independent set of vertices of $G(x, L)$ contains $v$. Under the assumption that $G(x, L)$ is connected, this means that $G(x, L)$ is a path of odd order and $v$ is one of the even numbered vertices, if we count along the path with the count starting with the number 1 (since the path has odd order, you can start from either end).

For each vertex $v \in V(G)$ we partition its supply, $L(v)$, as follows:

$$B(v) = \{x \in L(v) : x \text{ is bad at } v\}$$

$$O(v) = \{x \in L(v) : G(x, L) \text{ is a path of odd order }\} \setminus B(v)$$

$$E(v) = L(v) \setminus (B(v) \cup O(v)).$$

If $G(x, L)$ is simply a vertex $v$ then $x \in O(v)$. If $G(x, L) = G$ then $x \in E(v)$. We now assume that $G, L$ and $k$ satisfy Hall’s condition, and that $|L(v)| \geq 2k$ for all $v \in V(G)$, and set about showing that there is an $(L, k)$-coloring, by the strategy described earlier.
If \( x \in B(v) \) then \( x \in O(u) \) for each neighbor \( u \) of \( v \). Therefore, if \( |O(v) \cup E(v)| < k \) then \( |O(u)| \geq |B(v)| > k \) for each neighbor \( u \) of \( v \), since \( |L(u)| \geq 2k \) for every vertex. It follows that there is a vertex \( v_0 \) such that \( |O(v_0) \cup E(v_0)| \geq k \). Let the vertices of \( G \) be \( v_0, v_1, \ldots, v_n \), one way or the other around the cycle.

Let \( X_0 \subseteq O(v_0) \cup E(v_0) \) be of size \( k \) and such that \( |X_0 \cap O(v_0)| \) is as large as possible. We intend to color \( v_0 \) with \( X_0 \). Define \( L' \) on \( G - v_0 \) by

\[
L'(v_i) = \begin{cases} 
L(v_i) \setminus X_0 & \text{if } i = 1, n - 1 \\
L(v_i) & \text{otherwise.}
\end{cases}
\]

We shall finish the proof that \( h^{(k)}(C_n) \leq 2k \) by showing that \( G - v_0, L' \) and \( k \) satisfy Hall’s condition.

Let \( H \) be an induced subgraph of \( G - v_0 \). To verify \((*)\) for \( H, L' \) and \( w = k \), it suffices to verify it for every connected component of \( H \), so we may as well consider \( H \) to be connected; that is, \( H \) is a path. If \( H \) is a single vertex \( v_i \), \( \sum_{x \in C} \alpha(H(x, L')) = |L'(v_i)| \), which is either at least \( 2k \), if \( 1 < i < n - 1 \), or is \( |L(v_i) \setminus X_0| \geq 2k - k = k \), if \( i = \{1, n - 1\} \). In any case, \( \sum_{x \in C} \alpha(H(x, L')) \geq k = k|V(H)| \), so \((*)\) holds and we assume that \( |V(H)| > 1 \).

If \( H \) is a path containing neither \( v_1 \) nor \( v_{n-1} \) then \((*)\) holds because \( L = L' \) on \( V(H) \) and \( G, L \) and \( k \) are assumed to satisfy Hall’s condition. Therefore, we need only consider the case that \( H \) is a path containing either \( v_1 \) or \( v_{n-1} \), or both.

Suppose that \( H \) contains \( v_1 \) but not \( v_{n-1} \). (Disposing of this case will also take care of the case when \( H \) contains \( v_{n-1} \) but not \( v_1 \).) Then, using self-explanatory notation for paths, \( H = (v_1, v_2, \ldots, v_t) \), for some \( t, 1 < t < n - 1 \). Since \( v_t, v_{t-2}, \ldots, v_{t-2r} \), where \( r = \lfloor \frac{t-1}{2} \rfloor \), are independent vertices in \( H \), we have that

\[
\sum_{x \in C} \alpha(H(x, L')) \geq \sum_{i=0}^{t} |L'(v_{t-2i})| \\
\geq 2rk + k \quad \text{if } t \text{ is odd} \\
2(r + 1)k \quad \text{if } t \text{ is even} \\
= tk = k|V(H)|.
\]

Now suppose that \( H = (v_1, v_2, \ldots, v_{n-1}) = G - v_0 \).

If \( x \in C \setminus X_0 \) then \( \alpha(H(x, L')) = \alpha(H(x, L)) \). Also, the definition of \( O(v_0) \) implies that, for every \( x \in O(v_0) \), \( \alpha(H(x, L')) = \alpha(H(x, L)) = \alpha(G(x, L)) - 1 \). If \( |O(v_0)| \geq k \) then \( X_0 \) is a subset of \( O(v_0) \) so we have that \( \alpha(H(x, L')) = \alpha(H(x, L)) \) for all \( x \), and we are done by the assumption that \( G, L \) and \( k \) satisfy Hall’s condition.

Therefore, we may assume that \( |O(v_0)| < k \) and, consequently, by the choice of \( X_0 \), that \( O(v_0) \subseteq X_0 \). As noted above, if \( x \in O(v_0) \), \( \alpha(H(x, L')) = \alpha(H(x, L)) = \alpha(G(x, L)) - 1 \). For \( x \in X_0 \setminus O(v_0) \subseteq E(v_0) \), we also have that
\[\alpha(H(x, L')) \geq \alpha(G(x, L)) - 1,\] by the definition of \(E(v_0)\) and of \(L'\). Thus \(\alpha(H(x, L')) \geq \alpha(G(x, L)) - 1\) for all \(x \in X_0\). If \(x \in C \setminus X_0\) then either \(x \notin L(v_0)\), in which case \(\alpha(H(x, L')) = \alpha(G(x, L))\), or \(x \in B(v_0) \cup (E(v_0) \setminus X_0)\), in which case \(\alpha(H(x, L')) = \alpha(G(x, L))\), as well. Consequently,

\[
\sum_{x \in C} \alpha(H(x, L')) = \sum_{x \in X_0} \alpha(H(x, L')) + \sum_{x \in C \setminus X_0} \alpha(H(x, L')) \\
\geq \sum_{x \in X_0} \alpha(G(x, L)) - |X_0| + \sum_{x \in C \setminus X_0} \alpha(G(x, L)) \\
= \sum_{x \in C} \alpha(G(x, L)) - k \\
\geq kn - k \\
= k|V(H)|.
\]

This completes the proof \(h^k(C_n) \leq 2k\).

Next we show that \(h^k(C_{2m+1}) = 2k\) for all \(k \geq 1\) and \(m \geq 2\) by exhibiting a color supply \(L\) satisfying Hall’s condition with \(G = C_{2m+1}\) and \(w = k\) and \(|L(v)| \geq 2k - 1\) for all \(v\), such that there is no \((L, k)\)-coloring of \(G\). Let \(v_0, v_1, \ldots, v_{2m}\) be the vertices of \(G\) around the cycle. For any positive integer \(z\), let \([z] = \{1, 2, \ldots, z\}\) and define \(L(v_0) = [2k - 1]\), \(L(v_1) = L(v_2m) = [2k]\) and \(L(v_j) = [2k] + (2k - 1)\) for \(2 \leq j \leq 2m - 1\), where \([z] + x = \{1 + x, 2 + x, \ldots, z + x\}\).

In any \((L, k)\)-coloring of \(G\), because the lists \(L(v_j)\), \(2 \leq j \leq 2m - 1\), have only \(2k\) elements, color \(2k\) would have to appear on either \(v_2\) or \(v_{2m-1}\). On the other hand, in every \((L, k)\)-coloring of \((v_1, v_0, v_{2m})\) color \(2k\) must be used on both \(v_1\) and \(v_{2m}\). Therefore, no such coloring of the cycle exists. It is straightforward to verify that there is an \((L, k)\)-coloring of \(G - v\) for every \(v \in V\); therefore (*) holds for every proper induced subgraph \(H\) of \(G\). Thus we need only to verify (*) for \(H = G\). The following are easily seen: for \(x \in [2k - 1]\), \(\alpha(G(x, L)) = 2\); \(\alpha(G(2k, L)) = m\) and for \(x \in [2k - 1] + 2k\), \(\alpha(G(x, L)) = m - 1\). Therefore,

\[
\sum_{x \in C} \alpha(G(x, L)) = 2(2k - 1) + m + (m - 1)(2k - 1) \\
= k(2m + 1) + k - 1 \\
\geq k(2m + 1) \\
= k|V(G)|.
\]

It may be worth pointing out that the list assignment given to show that \(h^k(C_{2m+1}) \geq 2k\) does not do the same job for \(C_{2m}, m \geq 2\), for the reason that there is an \((L, k)\)-coloring of \(C_{2m}\), if \(L\) is defined as above.
To finish the proof of the theorem, we suppose that \( m \geq \lceil \frac{k+2}{3} \rceil \), so that \( n = 2m > k+1 \), and we give a list assignment \( L \) to \( G = C_{2m} \) such that \( G, L \) and \( k \) satisfy Hall’s condition, and \( |L(v)| \geq 2k-1 \) for all \( v \in V \), yet there is no \((L,k)\)-coloring of \( G \). Let the vertices of \( G \) be \( v_0, v_1, \ldots, v_{2m-1} \) around the cycle.

**Case 1.** \( m \leq k \): set \( L(v_0) = [2k], \ L(v_j) = [2k-1] + j, \ 1 \leq j \leq 2m-3, \ L(v_{2m-2}) = [2k] + (2m - 3) \) and \( L(v_{2m-1}) = \{1,3,5,\ldots,2m-3\} \cup \{2k + 1, 2k + 3, 2k + 5, \ldots, 2k + 2m - 3\} \cup ([k - 1] + (2k + 2m - 3)). \)

**Case 2.** \( m \geq k + 1 \): set \( L(v_0) = [2k], \ L(v_j) = [2k-1] + j, \ 1 \leq j \leq 2k - 1, \ L(v_j) = [2k] + (2k - 1), \ 2k \leq j \leq 2m - 2 \) and \( L(v_{2m-1}) = \{1,3,5,\ldots,2k-1\} \cup \{2k + 1, 2k + 3, 2k + 5, \ldots, 4k - 1\} \cup ([k - 1] + (4k - 1)). \)

First note that all lists have size at least \( 2k - 1 \). The only unobvious case is that of \( L(v_{2m-1}) \) in Case 1, where we have \( |L(v_{2m-1})| = m - 1 + m - 1 + k - 1 = 2m + k - 3 \geq k + 2 + k - 3 = 2k - 1 \).

Next we shall see that there is no \((L,k)\)-coloring of \( G \). If \( \varphi \) were such a coloring, then, because \( |L(v_j) \cup L(v_{j+1})| = 2k \), \( 0 \leq j \leq 2m - 3 \), it must be that \( \varphi(v_j) = L(v_j) \setminus \varphi(v_{j+1}) \), \( 0 \leq j \leq 2m - 3 \) and \( \varphi(v_j) = L(v_j) \setminus \varphi(v_{j+1}) \), \( 1 \leq j \leq 2m - 2 \). Related observations: for \( 0 \leq j \leq 2m - 4 \) in Case 1 and for \( 0 \leq j \leq 2k - 2 \) in Case 2, it must be that \( j + 1 \in \varphi(v_j) \).

Otherwise, \( \varphi(v_j) \subseteq L(v_{j+1}) \) and we would have \( |\varphi(v_{j+1})| = |L(v_{j+1}) \setminus \varphi(v_j)| = 2k - 1 - k \).

By the same argument, slightly modified, \( 2m - 2 \in \varphi(v_{2m-3}) \) in Case 1 and \( 2k \in \varphi(v_{2k-1}) \) in Case 2. Similarly, for \( 2 \leq j \leq 2m - 2 \) in Case 1 and for \( 2 \leq j \leq 2k \) in Case 2, it must be that \( 2k - 1 + j \in \varphi(v_j) \).

Using these observations with \( \varphi(v_j) = L(v_j) \setminus \varphi(v_{j+1}) \) for various values of \( j \), it can be seen that, in Case 1, \( \varphi(v_0) \) must contain \( 1,3,\ldots,2m - 3 \) and \( \varphi(v_{2m-2}) \) must contain \( 2m + 2k - 3, 2m + 2k - 5, \ldots, 2k + 1 \). But this leaves only \( k - 1 \) colors in \( L(v_{2m-1}) \) eligible for \( \varphi(v_{2m-1}) \). So there is no such \( \varphi \) in Case 1. In Case 2, something similar happens: \( \varphi(v_0) \) must contain \( 1,3,\ldots,2k - 1 \) and \( \varphi(v_{2m-2}) \) must contain \( 4k - 1, 4k - 3, \ldots, 2k + 1 \), which leaves only \( k - 1 \) elements in \( L(v_{2m-1}) \) eligible for \( \varphi(v_{2m-1}) \).

To show that \( G, L \) and \( w = k \) satisfy Hall’s condition is straightforward, but enough of a chore that we shall bear some of the burden here. First we show that \( G - v \) has an \((L,k)\)-coloring for each \( v \in V \). From the discussion of the non-existence of \( \varphi \), above, it is clear that this holds for \( v = v_{2m-1} \), in both cases, but for other \( v \in V \) there is work to be done. We start by explicitly defining a “near” \( k \)-fold coloring of \( G \) from the list assignment \( L \), the very \( \varphi \) whose values on \( v_0, v_1, \ldots, v_{2m-2} \) are forced in Case 2 and partially in Case 1. For each \( v \in V \), let \( odd(v) \) and \( even(v) \) denote respectively the subsets of odd and even elements in \( L(v) \), and for \( 0 \leq j \leq 2m - 2 \),

\[
\varphi(v_j) = \begin{cases} 
odd(v_j) & \text{if } j \text{ is even,} \\
_{
\text{even}(v_j) & \text{if } j \text{ is odd;}
\end{cases}
\]

further,

\[
\varphi(v_{2m-1}) = \begin{cases} 
[k-1] + (2k + 2m - 3) & \text{in Case 1} \\
[k-1] + (4k - 1) & \text{in Case 2.}
\end{cases}
\]
Finally, for \( \alpha \phi(v_{2m-1}) \) is an \((L, k)\)-coloring of that graph. Further, \( \varphi \) can be modified to an \((L, k)\)-coloring of \( G - v_0 \) by adding 1 to \( \varphi(v_{2m-1}) \) and to an \((L, k)\)-coloring of \( G - v_{2m-2} \) by adding \( 2k + 2m - 3 \) in Case 1, and \( 4k - 1 \) in Case 2, to \( \varphi(v_{2m-1}) \). For \( i \in [2m - 3] \) we modify \( \varphi \) to an \((L, k)\)-coloring \( \varphi^* \) on \( G - v_1 \) as follows:

In Case 1, if \( i \) is even, then for \( 0 \leq j < i \),

\[
\varphi^*(v_j) = \begin{cases} 
(\varphi(v_j)\setminus\{i + 1\}) \cup \{i\} & \text{if } j \text{ is even} \\
(\varphi(v_j)\setminus\{i\}) \cup \{i + 1\} & \text{if } j \text{ is odd},
\end{cases}
\]

and for \( i < j \leq 2m - 2 \), \( \varphi^*(v_j) = \varphi(v_j) \) and \( \varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i + 1\} \); if \( i \) is odd then for \( 0 \leq j < i \),

\[
\varphi^*(v_j) = \begin{cases} 
(\varphi(v_j)\setminus\{i + 1\}) \cup \{i\} & \text{if } j \text{ is odd} \\
(\varphi(v_j)\setminus\{i\}) \cup \{i + 1\} & \text{if } j \text{ is even},
\end{cases}
\]

and for \( i < j \leq 2m - 2 \), \( \varphi^*(v_j) = \varphi(v_j) \) and \( \varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i\} \).

In Case 2, for \( i \in [2k - 1] \), if \( i \) is even then for \( 0 \leq j < i \),

\[
\varphi^*(v_j) = \begin{cases} 
(\varphi(v_j)\setminus\{i + 1\}) \cup \{i\} & \text{if } j \text{ is even} \\
(\varphi(v_j)\setminus\{i\}) \cup \{i + 1\} & \text{if } j \text{ is odd},
\end{cases}
\]

for \( i < j \leq 2m - 2 \), \( \varphi^*(v_j) = \varphi(v_j) \), and \( \varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i + 1\} \); if \( i \) is odd then for \( 0 \leq j < i \),

\[
\varphi^*(v_j) = \begin{cases} 
(\varphi(v_j)\setminus\{i + 1\}) \cup \{i\} & \text{if } j \text{ is odd} \\
(\varphi(v_j)\setminus\{i\}) \cup \{i + 1\} & \text{if } j \text{ is even},
\end{cases}
\]

for \( i < j \leq 2m - 2 \), \( \varphi^*(v_j) = \varphi(v_j) \), and \( \varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{i\} \). If \( 2k \leq i \leq 2m - 2 \), for \( 0 \leq j < i \), set \( \varphi^*(v_j) = \varphi(v_j) \), and for \( i < j \leq 2m - 2 \),

\[
\varphi^*(v_j) = \begin{cases} 
(\varphi(v_j)\setminus\{4k - 2\}) \cup \{4k - 1\} & \text{if } j \text{ is odd} \\
(\varphi(v_j)\setminus\{4k - 1\}) \cup \{4k - 2\} & \text{if } j \text{ is even},
\end{cases}
\]

and \( \varphi^*(v_{2m-1}) = \varphi(v_{2m-1}) \cup \{4k - 1\} \).

The proof will be complete when we verify that (*) holds for \( G \), that is \( \sum_{x \in C} \alpha(G(x, L)) \geq nk = 2mk \).

**Case 1.** For \( 1 \leq x \leq 2m - 3 \), \( G(x, L) \) is a path of order \( x + 1 \) if \( x \) is odd and \( x \) if \( x \) is even, so \( \alpha(G(x, L)) = \left\lceil \frac{x}{2} \right\rceil \). For \( 2m - 2 \leq x \leq 2k \), \( G(x, L) = G - v_{2m-1} \), so \( \alpha(G(x, L)) = m \). For \( 2k + 1 \leq x \leq 2m + 2k - 3 \), \( G(x, L) \) is a path of order \( 2m + 2k - x - 2 \) if \( x \) is even, and of order \( 2m + 2k - x - 1 \) if \( x \) is odd, so \( \alpha(G(x, L)) = m + k - 1 - \left\lceil \frac{x}{2} \right\rceil \). Finally, for \( x \in [k - 1] + (2k + 2m - 3) \), \( \alpha(G(x, L)) = 1 \). Therefore,
\[ \sum_{x \in C} \alpha(G(x, L)) = \sum_{x=1}^{\frac{2m-3}{2}} \left\lfloor \frac{x}{2} \right\rfloor + \sum_{x=2m-2}^{2k} m + \sum_{x=2k+1}^{2m+2k-3} \left( m + k - 1 - \left\lfloor \frac{x}{2} \right\rfloor \right) + (k - 1) \]

\[ = (m - 1)^2 + m(2k - 2m + 3) + (m + k - 1)(2m - 3) \]

\[ - ((m + k - 1)(m + k - 2) - k^2) + k - 1 \]

\[ = 2mk + k - m + 1 \]

\[ \geq 2mk \]

\[ = kn. \]

**Case 2.** For \( 1 \leq x \leq 2k - 1 \), \( G(x, L) \) is a path of order \( x + 1 \) if \( x \) is odd and \( x \) if \( x \) is even; thus \( \alpha(G(x, L)) = \left\lfloor \frac{x}{2} \right\rfloor \). \( G(2k, L) = G - v_{2m-1} \), so \( \alpha(G(2k, L)) = m \). For \( x = 2k + r, \ 1 \leq r \leq 2k - 2 \), \( G(x, L) \) is a path of order \( 2m - 2 - r \) if \( r \) (and thus \( x \)) is even, and of order \( 2m - 1 - r \) if \( r \) (and thus \( x \)) is odd; \( \alpha(G(x, L)) = m - 1 - \left\lfloor \frac{r}{2} \right\rfloor \). Clearly \( \alpha(G(4k-1, L)) = m - k \) and \( \alpha(G(x, L)) = 1 \) for \( x \in [k-1] + (4k-1) \). Therefore,

\[ \sum_{x \in C} \alpha(G(x, L)) = \sum_{x=1}^{\frac{2k-1}{2}} \left\lfloor \frac{x}{2} \right\rfloor + \sum_{r=1}^{2k-2} \left( m - 1 - \left\lfloor \frac{r}{2} \right\rfloor \right) + (m - k) + (k - 1) \]

\[ = k^2 + m + (m - 1)(2k - 2) - (k - 1)^2 + (m - 1) \]

\[ = 2mk \]

\[ = kn. \]

\[ \square \]

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**References**


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