Signed reinforcement numbers of certain graphs

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Communicated by: T.W. Haynes
Received 07 November 2011; accepted 01 March 2012

Abstract

Let $G$ be a graph with vertex set $V(G)$. A function $f: V(G) \rightarrow \{-1, 1\}$ is a signed dominating function of $G$ if, for each vertex of $G$, the sum of the values of its neighbors and itself is positive. The signed domination number of a graph $G$, denoted $\gamma_s(G)$, is the minimum value of $\sum_{v \in V(G)} f(v)$ over all the signed dominating functions $f$ of $G$. The signed reinforcement number of $G$, denoted $R_s(G)$, is defined to be the minimum cardinality $|S|$ of a set $S$ of edges such that $\gamma_s(G + S) < \gamma_s(G)$. In this paper, we initialize the study of signed reinforcement number and determine the exact values of $R_s(G)$ for several classes of graphs.

Keywords: Signed domination, signed reinforcement number.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The open neighborhood, the closed neighborhood and the degree of $v \in V(G)$ are defined by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, $N_G[v] = N_G(v) \cup \{v\}$ and $d_G(v) = |N_G(v)|$, respectively. For $S \subseteq V(G)$, $N_G(S)$ is defined to be the union of the open neighborhoods $N_G(v)$ for all $v \in S$ and $N_G[S] = N_G(S) \cup S$. Let $\Delta(G)$ denote the maximum degree of a graph $G$. A vertex of degree one in $G$ is called a leaf; a support vertex of $G$ is a vertex adjacent with a leaf of $G$. Let $L(G)$ and $S(G)$ denote the set of leaves of $G$ and the set of support vertices of $G$, respectively. For two sets $A, B \subseteq V(G)$, let $E(A, B) = \{e = xy \mid x \in A, y \in B\}$ and $e(A, B) = |E(A, B)|$.

Let $G = (V, E)$ be a graph and $f: V \rightarrow R$ is a real-valued function on $V$. The weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$. For $S \subseteq V$, define $f(S) = \sum_{v \in S} f(v)$. Then $\omega(f) = f(V)$.

*The work was supported by NNSF of China and the Fundamental Research Funds for the Central Universities.

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For any \( v \in V \), let \( f[v] = f(N[v]) \) for notation convenience. A function \( f: V \rightarrow \{-1, 1\} \) is called a signed dominating function (abbreviated by SDF) if \( f[v] \geq 1 \) for all \( v \in V \). The signed domination number of \( G \) is \( \gamma_s(G) = \min \{ \omega(f) \mid f \text{ is a SDF of } G \} \). A \( \gamma_s(G) \)-function is a signed dominating function of \( G \) of weight \( \gamma_s(G) \). Signed domination was first introduced by Dunbar et al. in [4] and further studied in [1, 3, 6, 7, 9, 14, 10, 11, 12, 13, 15].

The reinforcement number of a graph \( G \) is a measurement of the stability of the domination in \( G \). The reinforcement number of a graph \( G \) is the smallest number of edges which must be added to \( G \) to decrease the domination number of \( G \) (the classic domination number of a graph \( G \) is the minimum cardinality of a subset \( D \) of \( V(G) \) such that for each \( v \in V(G), N[v] \cap D \neq \emptyset \)). The definition was first introduced by Kok and Mynhardt [8]. During the past twenty years, the reinforcement number associated with domination parameters were studied in literatures, for example, Ghoshal et al. [5] defined and studied the reinforcement number associated with the strong domination number; Gayla et al. [2] studied the reinforcement number associated with the fractional domination number.

In this paper, we define the signed reinforcement number of a graph \( G \), denoted \( R_s(G) \), to be the minimum cardinality of a set \( S \) of edges in the complement graph \( G^c \) of \( G \) such that \( \gamma_s(G + S) < \gamma_s(G) \). A minimum edge set \( S \subseteq E(G^c) \) with \( \gamma_s(G + S) < \gamma_s(G) \) is called a signed reinforcement set of \( G \). Note that the signed reinforcement set of a graph \( G \) may not exist, for example, for \( K_n \), the complete graph on \( n \) vertices or \( C_4 \), the cycle on 4 vertices. So if the signed reinforcement set of a graph \( G \) doesn’t exist, we define \( R_s(G) = 0 \).

The paper is organized as follows. Section 2 gives some lemmas about signed domination numbers and signed reinforcement numbers. Sections 3 and 4 determine the exact values of the signed reinforcement numbers of paths, cycles and wheels. Section 5 gives a sharp bound of the signed reinforcement number of trees.

2. Lemmas

In this section, we will give some useful lemmas about signed dominating functions of a graph \( G \). Let \( K_n, P_n \) and \( C_n \) denote a complete graph, a path and a cycle on \( n \) vertices, respectively. The following lemmas are given in [4] and the proof of them can be found in [4].

**Lemma 2.1.** [4] A signed dominating function \( f \) on a graph \( G \) is minimal if and only if for every vertex \( v \in V \) with \( f(v) = 1 \), there exists a vertex \( u \in N[v] \) with \( f[u] \in \{1, 2\} \).

**Lemma 2.2.** [4] If \( f \) is a signed dominating function of a graph \( G \), then \( f(v) = 1 \) for any \( v \in L(G) \cup S(G) \).

**Lemma 2.3.** [4] Let \( G \) be a graph on \( n \) vertices. Then \( \gamma_s(G) = n \) if and only if \( V(G) = L(G) \cup S(G) \).

**Lemma 2.4.** [4] If \( G \) has more than three vertices and maximum degree \( \Delta \leq 3 \), then \( \gamma_s(G) \geq \frac{n}{3} \).
The following lemma gives a lower bound for the signed domination number of a graph \( G \) with precisely one vertex with maximum degree four.

**Lemma 2.5.** Let \( G \) be a graph with order \( n \) and maximum degree four. If \( G \) has precisely one vertex with maximum degree four, then \( \gamma_s(G) \geq \frac{n-2}{3} \).

**Proof.** Let \( f \) be a \( \gamma_s \)-function and let \( P \) and \( M \) be the reverse images of +1 and -1 under \( f \). Then \(|P| + |M| = n \) and \( \gamma_s(G) = |P| - |M| \).

If \( M = \emptyset \), then \( \gamma_s(G) = n > \frac{n-2}{3} \).

If \( M \neq \emptyset \), we evaluate the number, \( e(M, P)_s \), of edges between \( P \) and \( M \) in \( G \).

For any \( v \in M \), to guarantee \( f[v] \geq 1 \), there exist at least two edges from \( v \) to \( P \), which means that \( e(M, P)_s \geq 2|M| \).

On the other hand, for each \( v \in P \), to guarantee \( f[v] \geq 1 \), \( |N(v) \cap M| \leq |N(v) \cap P| \).

Hence there are at most \( \frac{d(v)}{2} \) edges from \( v \) to \( M \). Since \( G \) has precisely one vertex with maximum degree four, \( e(M, P)_s \leq |P| - 1 + 2 = |P| + 1 \).

Hence, \( 2|M| \leq e(M, P) \leq |P| + 1 \). Combine with \(|P| + |M| = n\), we have \(|M| \leq \frac{n+1}{3}\) and \(|P| \geq \frac{2n-1}{3}\). So, \( \gamma_s(G) = |P| - |M| \geq \frac{2n-1}{3} - \frac{n+1}{3} = \frac{n-2}{3} \). \( \square \)

The signed domination numbers of paths, cycles and stars were given in \([4]\).

**Lemma 2.6.** \([4]\)

1. \( \gamma_s(K_{1,n-1}) = n, n \geq 2; \)
2. \( \gamma_s(P_n) = n - 2\lfloor \frac{n-2}{3} \rfloor, n \geq 2; \)
3. \( \gamma_s(C_n) = n - 2\lfloor \frac{n}{2} \rfloor, n \geq 3. \)

**Lemma 2.7.** Let \( G \) be a connected graph with \(|V(G)| \geq 3\). If \( \gamma_s(G) = |V(G)| \), then \( R_s(G) = 1 \).

**Proof.** Since \( \gamma_s(G) = |V(G)| \), by Lemma 2.3, \( V(G) = L(G) \cup S(G) \). Let \( f \) be a \( \gamma_s \)-function.

Since \( \gamma_s(G) = |V(G)| \), \( f \equiv 1 \). Since \(|V(G)| \geq 3 \), \(|L(G)| \geq 2 \). Let \( u, v \in L(G) \) and \( w \) be the support vertex of \( u \). Since \( f \equiv 1 \), \( f[w] \geq 3 \). Then if we replace the value 1 by –1 on \( u \) and adding the edge \( uv \) to \( G \), then the reduced function is a SDF of \( G + uv \) with weight \(|V(G)| - 2 < \gamma_s(G) \). So \( R_s(G) = 1 \). \( \square \)

**Lemma 2.8.** For any graph \( G \), if \( \gamma_s(G + A) < \gamma_s(G) \) for some set \( A \subseteq E(G^c) \), then \( \gamma_s(G + A) \leq \gamma_s(G) - 2 \).

**Proof.** Let \( f \) and \( g \) be minimum signed dominating functions of \( G + A \) and \( G \), respectively, and let \( f^{-1}(a) \) and \( g^{-1}(a) \) denote the reversed imagines of \( a \) under \( f \) and \( g \). Since \( \gamma_s(G + A) < \gamma_s(G) \), \( |f^{-1}(1)| \leq |g^{-1}(1)| - 1 \) (equivalently, \(|f^{-1}(-1)| \geq |g^{-1}(-1)| + 1\)). Hence \( \gamma_s(G + A) = |f^{-1}(1)| - |f^{-1}(-1)| \leq |g^{-1}(1)| - |g^{-1}(-1)| - 2 = \gamma_s(G) - 2 \). \( \square \)
3. The signed reinforcement numbers of stars, paths and cycles

Since \( V(K_{1,n-1}) = L(K_{1,n-1}) \cup S(K_{1,n-1}) \), by Lemmas 2.3 and 2.7, \( R_s(K_{1,n-1}) = 1 \) if \( n \geq 3 \). Hence, we have the following observation.

**Observation 3.1.** Let \( n \geq 3 \). Then \( R_s(K_{1,n-1}) = 1 \).

**Theorem 3.2.** For \( n \geq 3 \),

\[
R_s(P_n) = \begin{cases} 
2, & n \equiv 2 \pmod{3} \\
1, & \text{otherwise.}
\end{cases}
\]

**Proof.** Denote \( V(P_n) = \{v_1, v_2, \cdots, v_n\} \).

If \( n = 3k \) or \( 3k + 1 \) for some integer \( k(\geq 1) \), then

\[
\gamma_s(P_{3k} + v_1v_{3k}) = \gamma_s(C_{3k}) = k < k + 2 = \gamma_s(P_{3k})
\]

and

\[
\gamma_s(P_{3k+1} + v_1v_{3k+1}) = \gamma_s(C_{3k+1}) = k + 1 < k + 3 = \gamma_s(P_{3k+1}).
\]

This implies that \( R_s(P_n) = 1 \) if \( n \not\equiv 2(\text{mod} 3) \).

If \( n = 3k + 2 \) for some integer \( k \geq 1 \), let \( G \) be the graph obtained from \( P_{3k+2} \) by adding two edges \( v_1v_3, v_3v_{3k+2} \). Now, define a function \( f \) as follows:

\[
f(v_i) = \begin{cases} 
-1, & i \equiv 1 \pmod{3} \\
1, & \text{otherwise}
\end{cases}
\]

It is an easy task to check that \( f[v_i] = 1 \) for every \( i \in [1, 3k + 2] \). So \( f \) is a SDF of \( G \). Hence, \( \gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(P_{3k+2}) \). Therefore, \( R_s(P_{3k+2}) \leq 2 \).

Now we show that \( R_s(P_{3k+2}) = 2 \). If there exists some edge \( e \not\in E(P_{3k+2}) \) such that \( \gamma_s(P_{3k+2} + e) < \gamma_s(P_{3k+2}) \), then, by Lemma 2.8, \( \gamma_s(P_{3k+2} + e) \leq \gamma_s(P_{3k+2}) - 2 = k \). Since \( \Delta(P_{3k+2} + e) \leq 3 \), by Lemma 2.4, \( \gamma_s(P_{3k+2} + e) \geq \lceil \frac{3k+2}{3} \rceil = k + 1 > k \geq \gamma_s(P_{3k+2} + e) \), a contradiction. \( \square \)

**Lemma 3.3.** Let \( n \geq 3 \) and \( n \equiv 0 \) or \( 1 \pmod{3} \). Then

\[
R_s(C_n) = \begin{cases} 
0, & n = 3, 4 \\
3, & n \geq 6.
\end{cases}
\]

**Proof.** Denote \( V(C_n) = \{v_0, v_1, \cdots, v_{n-1}\} \) and \( E(C_n) = \{v_i v_{i+1} \mid i = 0, 1, \cdots, n-1\} \), where the "+" is under modulo \( n \). If \( n = 3 \), then \( C_n = K_3 \) and \( R_s(C_n) = R_s(K_3) = 0 \).

If \( n = 4 \), we can check that \( \gamma_s(C_4) = 2 = \gamma_s(C_4 + v_1v_3) = \gamma_s(C_4 + v_0v_2) = \gamma_s(C_4 + \{v_1v_3, v_0v_2\}) \) and hence \( R_s(C_4) = 0 \).
If \( n \geq 6 \), let \( G \) be the graph obtained from \( C_n \) by adding three edges \( v_1v_3, v_1v_5 \) and \( v_3v_5 \) and define a function \( f : V(G) \rightarrow \{-1, 1\} \) by

\[
f(v_i) = \begin{cases} 
-1, & i = 2, 4 \text{ or } 3j \text{ for } j \in [2, \|\frac{n}{3}\|] \\
1, & \text{otherwise.}
\end{cases}
\]

It is an easy task to check that \( f[v_i] \geq 1 \) for any \( i \in [1, n] \). So \( f \) is a SDF of \( G \) and hence \( \gamma_s(G) \leq f(V(G)) = n - 2(\|\frac{n}{3}\| + 1) = n - 2\|\frac{n}{3}\| - 2 < n - 2\|\frac{n}{3}\| = \gamma_s(C_n) \) (Lemma 2.6 (3)). So we have \( R_s(C_n) \leq 3 \).

Next we will show that \( R_s(C_n) \geq 3 \) and so the result follows. Suppose to the contrary that there exist two edges \( e_1, e_2 \notin E(C_n) \) such that \( \gamma_s(C_n + \{e_1, e_2\}) < n - 2\|\frac{n}{3}\| = \gamma_s(C_n) \). By Lemma 2.8, \( \gamma_s(C_n + \{e_1, e_2\}) \leq \gamma_s(C_n) - 2 \).

If \( e_1, e_2 \) are independent, then \( \Delta(C_n + \{e_1, e_2\}) \leq 3 \). By Lemma 2.4, \( \gamma_s(C_n + \{e_1, e_2\}) \geq \|\frac{n}{3}\| = n - 2\|\frac{n}{3}\| = \gamma_s(C_n) \), a contradiction.

If \( e_1, e_2 \) have a common end, then \( C_n + \{e_1, e_2\} \) has precisely one vertex with maximum degree four. By Lemma 2.5, \( \gamma_s(C_n + \{e_1, e_2\}) \geq \|\frac{n-2}{3}\| \geq n - 2\|\frac{n}{3}\| - 1 = \gamma_s(C_n) - 1 \), a contradiction too.

**Lemma 3.4.** If \( n \equiv 2 \mod 3 \) and \( n \geq 5 \), then \( R_s(C_n) = 2 \).

**Proof.** Let \( V(C_n) \) and \( E(C_n) \) be defined the same as in the former proof and let \( G \) be the graph obtained by adding two edges \( v_1v_3 \) and \( v_3v_5 \). Suppose \( n = 3k + 2 \) \( (k \geq 1) \). Define a function \( f : V(G) \rightarrow \{-1, 1\} \) by

\[
f(v_i) = \begin{cases} 
-1, & i = 2 \text{ or } 3j + 1 \text{ for } j \in [1, k] \\
1, & \text{otherwise.}
\end{cases}
\]

It is an easy task to check that \( f[v_i] \geq 1 \) for any \( i \in [1, n] \). So \( f \) is a SDF of \( G \) and \( \gamma_s(G) \leq f(V(G)) = k < k + 2 = \gamma_s(C_n) \). Hence \( R_s(C_n) \leq 2 \).

If we can show that \( R_s(C_n) \geq 2 \), then the result follows. Suppose that there exists some edge \( e \notin E(C_n) \) such that \( \gamma_s(C_n + e) < \gamma_s(C_n) \). By Lemma 2.8, \( \gamma_s(C_n + e) \leq \gamma_s(C_n) - 2 = k \). Since \( \Delta(C_n + e) = 3 \), \( \gamma_s(C_n + e) \geq \|\frac{k}{3}\| = k + 1 \) by Lemma 2.4, a contradiction with \( \gamma_s(C_n + e) \leq k \).

From the above two lemmas, we have

**Theorem 3.5.** Let \( n \geq 3 \). Then

\[
R_s(C_n) = \begin{cases} 
0, & n = 3, 4 \\
3, & n \equiv 0 \text{ or } 1 \mod 3 \text{ and } n \geq 6 \\
2, & n \equiv 2 \mod 3.
\end{cases}
\]
4. Wheels

A wheel is a graph obtained from a cycle by adding a new vertex such that it is adjacent with each vertex of the cycle. Let \( W_n = \{w\} \cup C_{n-1} \) denote a wheel obtained from a cycle \( C_{n-1} \) and a new vertex \( w \), called the central vertex of \( W_n \). In the following, we denote \( V(C_{n-1}) = \{v_0, v_1, \cdots, v_{n-2}\} \) and \( E(W_n) = \{wv_i, v_iv_{i+1}, i = 0, 1, \cdots, n-2\} \), where the sum is taken modulo \( n - 1 \).

First we determine the signed domination number of \( W_n \).

**Lemma 4.1.** For \( n \geq 4 \), \( \gamma_s(W_n) = n - 2\left\lfloor \frac{n-1}{3} \right\rfloor \).

**Proof.** Since we can extend a SDF of \( C_{n-1} \) to be a SDF of \( W_n \) by assigning 1 to the central vertex \( w \), \( \gamma_s(W_n) \leq \gamma_s(C_{n-1}) + 1 = n - 1 - 2\left\lfloor \frac{n-1}{3} \right\rfloor + 1 = n - 2\left\lfloor \frac{n-1}{3} \right\rfloor 

In the following, we show that \( \gamma_s(W_n) \geq n - 2\left\lfloor \frac{n-1}{3} \right\rfloor \). Let \( f \) be a minimum SDF of \( W_n \) and let \( P \) and \( M \) be the set of reverse imagines of 1 and \(-1\) under \( f \), respectively. We claim that \( f(w) = 1 \), equivalently, \( w \in P \). If \( f(w) = -1 \), to guarantee \( f(v_i) \geq 1 \) for any \( i = 0, \cdots, n-2 \), \( f(v_i) = 1 \) since \( d(v_i) = 3 \). This means that \( \gamma_s(W_n) = n - 1 > n - 2\left\lfloor \frac{n-1}{3} \right\rfloor \), a contradiction.

Since \( f[v_i] = f(w) + f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 1 \), \( f(v_{i-1}) + f(v_i) + f(v_{i+1}) \geq 0 \). Hence at most one of three consecutive vertices on \( C_{n-1} \) is assigned \(-1\) by \( f \). This implies that \( |M| \leq \frac{n-1}{2} \). So \( \gamma_s(W_n) = n - 2|M| \geq n - 2\left\lfloor \frac{n-1}{3} \right\rfloor \). \( \square \)

**Theorem 4.2.**

1. \( R_s(W_4) = 0 \).

2. If \( n \geq 5 \),

\[
R_s(W_n) = \begin{cases} 
2, & n \equiv 1 \pmod{3} \\
1, & \text{otherwise}
\end{cases}
\]

**Proof.** (1) It follows directly from \( W_4 = K_4 \) and \( R_s(K_n) = 0 \) for any \( n \geq 2 \).

(2) If \( n = 3k \) (\( k \geq 2 \)) or \( n = 3k+2 \) (\( k \geq 1 \)), then, by Lemma 4.1, \( \gamma_s(W_n) = n - 2\left\lfloor \frac{n-1}{3} \right\rfloor = k + 2 \). Now, we add an edge \( v_0v_2 \) to \( W_n \) and define a function \( g : V(W_n + v_0v_2) \rightarrow \{-1, 1\} \) as follows:

\[
g(x) = \begin{cases} 
-1, & \text{if } x = v_i \text{ and } i = 1, n - 2 \text{ or } 3j \text{ for } j \in [1, \left\lfloor \frac{n}{3} \right\rfloor - 2] \\
1, & \text{otherwise}
\end{cases}
\]

It is an easy task to check that \( g \) is a SDF of \( W_n + v_0v_2 \). Hence \( \gamma_s(W_n + v_0v_2) \leq g(V(W_n)) = n - 2\left\lfloor \frac{n}{3} \right\rfloor = k < k + 2 = \gamma_s(W_{3k}) \). So \( R_s(W_n) = 1 \).

If \( n = 3k + 1 \) (\( k \geq 2 \)), then, by Lemma 4.1, \( \gamma_s(W_{3k+1}) = 3k + 1 - 2\left\lfloor \frac{3k+1-1}{3} \right\rfloor = k + 1 \). Then we can add two edges \( v_0v_2, v_2v_4 \) to \( W_n \) and define a SDF \( g \) of \( W_n + \{v_0v_2, v_2v_4\} \) as
follows:

$$g(x) = \begin{cases} 
-1, & \text{if } x = v_i \text{ and } i = 1, 3 \text{ or } 3i - 1 \text{ for } i \in [2, k] \\
1, & \text{otherwise}
\end{cases}$$

Hence

$$\gamma_s(W_{3k+1} + \{v_0v_2, v_2v_4\}) \leq g(V(W_n)) = 3k + 1 - 2(k + 1) = k - 1 < k + 1 = \gamma_s(W_{3k+1}).$$

So $$R_s(W_{3k+1}) \leq 2.$$ 

In the following, we will prove that $$R_s(W_{3k+1}) \geq 2.$$ Suppose to the contrary that there exists an edge $$e \notin E(W_n)$$ such that $$\gamma_s(W_{3k+1} + e) < \gamma_s(W_{3k+1}) = k + 1.$$ 

Let $$\phi$$ be a minimum SDF of $$W_{3k+1} + e$$ and let $$P$$ and $$M$$ be the reverse imaginaries of 1 and −1 under $$\phi$$, respectively. Then,

$$\begin{cases} 
|P| + |M| = 3k + 1 \\
|P| - |M| = \gamma_s(W_{3k+1} + e) \leq k
\end{cases}$$

Since $$|M|$$ and $$|P|$$ are integers, the equation array implies that

$$\begin{cases} 
|M| \geq k + 1 \\
|P| \leq 2k
\end{cases}$$

With a same reason with $$f(w) = 1$$ in the proof of Lemma 4.1, $$\phi(w) = 1.$$ Then $$M \subseteq V(C_{3k}).$$ Since $$|M| \geq k + 1,$$ there are three consecutive vertices $$v_{i-1}, v_i, v_{i+1}$$ on $$C_{3k}$$ such that two of them are in $$M.$$ 

If the two members of $$\{v_{i-1}, v_i, v_{i+1}\} \cap M$$ are consecutive on $$C_{3k},$$ without loss of generality, suppose $$v_{i-1}, v_i \in M.$$ Then, to guarantee that $$\phi[v_{i-1}] \geq 1, \phi[v_i] \geq 1, d(v_{i-1}) \geq 4$$ and $$d(v_i) \geq 4.$$ This is impossible since $$v_{i-1}, v_i$$ can not be the two ends of the new adding edge $$e.$$ Hence we must have $$v_{i-1}, v_{i+1} \in M.$$ 

To guarantee $$\phi[v_i] \geq 1, d(v_i) = 4$$ and $$\phi(v_i) = 1.$$ This means that $$v_i$$ must be an end of $$e.$$ Suppose $$e = v_i v_m.$$ Then $$\phi(v_m) = 1.$$ Now we compute the number of edges between $$M$$ and $$P - \{w\}$$ with two methods. Let $$P' = P - \{w\}.$$ 

Since, for each $$x \in M,$$ $$d(x) = 3$$ and $$\phi(x) = -1 + \phi(w) + \phi(N(x) \setminus \{w\}) \geq 1, \phi(N(x) \setminus \{w\}) \geq 1.$$ So, there are two edges from $$x$$ to the vertices in $$P',$$ this means $$e(x, P') = 2.$$ Hence $$e(M, P') = 2|M| \geq 2(k + 1).$$ 

Since, for each $$x \in P - \{w, v_i, v_m\},$$ $$d(x) = 3$$ and $$\phi(x) = 1 + 1 + \phi(N(x) \setminus \{w\}) \geq 1, \phi(N(x) \setminus \{w\}) \geq -1.$$ Hence there is at most one edge from $$x$$ to the vertices in $$M,$$ which means that $$e(x, M) \leq 1$$ for each $$x \in V(P - \{w, v_i, v_m\}).$$ For $$v_i$$ and $$v_m,$$ there are at most 2 edges from $$v_i$$ or $$v_m$$ to vertices in $$M.$$ So,

$$e(P', M) \leq |P - \{w, v_i, v_m\}| + 4 \leq 2k - 3 + 4 = 2k + 1 < 2(k + 1) \leq e(M, P'),$$

a contradiction. □
5. Trees

Lemma 5.1. For any tree $T$ with order $n \geq 3$, $R_s(T) \leq 3$.

Proof. If $\gamma_s(T) = n$, then $R_s(T) = 1 < 3$ by Lemma 2.7.

Now suppose $\gamma_s(G) < n$. Then $T \neq K_{1,n-1}$. Hence there exist two leaves $u_1, v_1$ such that they have two different support vertices $u_2$ and $v_2$, respectively.

If $L(T) = \{u_1, v_1\}$, then $T = P_n$ and so $R_s(T) \leq 2$ by Theorem 3.2.

If $L(T) \neq \{u_1, v_1\}$, let $w_1$ be another leaf of $T$. Then there is at least one of $u_2, v_2$ which is not adjacent with $w_1$. Without loss of generality, assume $u_2w_1 \notin E(T)$. Let $f$ be a minimum SDF of $T$. By Lemma 2.2, $f(u_i) = f(v_i) = 1, i = 1, 2$ and $f(w_1) = 1$. Let $S = \{u_1v_1, u_2w_1\}$ if $u_2v_2 \in E(G)$ and $S = \{u_1v_1, u_2v_2, u_2w_1\}$ if $u_2v_2 \notin E(G)$. We can easily modify $f$ to be a SDF $g$ of $T + S$ as follows.

\[
g(x) = \begin{cases} 
-1, & x = u_1 \\
f(x), & x \in V(T) - \{u_1\}
\end{cases}
\]

Then $\gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2 < \gamma_s(T)$ implies that $R_s(T) \leq 3$. □

In fact, the upper bound of the signed reinforcement number of trees given here is not sharp. In the following, we will give a sharp bound for $R_s(T)$.

Lemma 5.2. Let $f$ be a minimum SDF of a tree $T$. If there exists a support vertex $v$ with $f[v] \geq 3$, then $R_s(T) = 1$.

Proof. Let $u, w$ be two leaves of $T$ such that at least one of them is adjacent with $v$ in $T$. Then $uw$ is the desired edge to guarantee that $\gamma_s(T + uw) < \gamma_s(T)$. □

Lemma 5.3. Let $f$ be a minimum SDF of a tree $T$. If there exists a support vertex $v$ with $f[v] \geq 2$, then $R_s(T) \leq 2$.

Proof. If $T$ is a star, then the result is clearly true. Now suppose $T$ is not a star. Then we can choose two leaves $u, w$ of $T$ such that $uw \in E(T)$ and $uw \notin E(T)$. Hence $uw, vw$ are two edges to guarantee that $\gamma_s(T + \{uw, vw\}) < \gamma_s(T)$. So, $R_s(T) \leq 2$. □

Lemma 5.4. Let $T'$ be a tree obtained from a tree $T$ ($|V(T)| \geq 3$) by adding an edge joining a leaf of $T$ with a leaf of a path $P_3$. Then $R_s(T') \leq R_s(T)$.

Proof. Let $v$ be a leaf of $T$ and let $u$ be its support vertex. Let $P_3 = x_1x_2x_3$ and let $T'$ be a tree obtained from $T \cup P_3$ by adding an edge $x_3v$. It is an easy task to check that $\gamma_s(T') = \gamma_s(T) + 1$. Now suppose $R_s(T) = r$ and $S$ is a set of edges with $|S| = r$ such that $\gamma_s(T + S) \leq \gamma_s(T) - 2$ (by Lemma 2.8). By Lemma 5.1, $r \leq 3$.

Let $f$ be a minimum SDF of $T + S$. Then $f(V(T)) = \gamma_s(T + S)$. 

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If \( v \) is not incident with any edge in \( S \), then \( v \) is a leaf of \( T+S \), too. Hence \( f(v) = f(u) = 1 \). We can easily extend \( f \) to be a SDF, say \( g \), of \( T'+S \) by defining \( g(x_1) = g(x_2) = 1 \) and \( g(x_3) = -1 \) and \( g(x) = f(x) \) for the other vertices. So \( \gamma_s(T'+S) \leq \gamma_s(T+S) + 1 \leq \gamma_s(T) - 2 + 1 = \gamma_s(T') - 2 \). This implies that \( R_s(T') \leq |S'| = r = R_s(T) \).

Now we suppose that \( v \) is incident with some edges, denoted \( vu_1, \ldots, vu_t \) in \( S \).

If \( f[v] \geq 2 \), then we can extend \( f \) to be a SDF of \( T'+S \) the same as the above case and so the result is valid. So we assume that \( f[v] = 1 \) in the following.

**Case 1.** \( f(v) = 1 \).

If \( f(u) = 1 \), then \( f(u_1) + \cdots + f(u_t) = -1 \). Let \( S' = (S - \{vu_1, \ldots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\} \). Then we can define a SDF \( g \) of \( T'+S' \) as follows:

\[
g(x) = \begin{cases} 
-1, & x = x_3 \\
1, & x = x_1, x_2 \\
f(x), & x \in V(T)
\end{cases}
\]

So \( \gamma_s(T'+S') \leq g(V(T'+S')) \leq \gamma_s(T+S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2 \). This implies that \( R_s(T') \leq |S'| = |S| = R_s(T) \).

If \( f(u) = -1 \), then \( f(u_1) + \cdots + f(u_t) = 1 \). Let \( S' = (S - \{vu_1, \ldots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\} \). We also can define a SDF \( g \) of \( T'+S' \) as follows:

\[
g(x) = \begin{cases} 
-1, & x = x_2 \\
1, & x = x_1, x_3 \\
f(x), & x \in V(T)
\end{cases}
\]

So \( \gamma_s(T'+S') \leq g(V(T'+S')) \leq \gamma_s(T+S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2 \) implies that \( R_s(T') \leq |S'| = |S| = R_s(T) \).

**Case 2.** \( f(v) = -1 \).

If \( f(u) = 1 \), then \( f(u_1) + \cdots + f(u_t) = 1 \). Let \( S' = (S - \{vu_1, \ldots, vu_t\}) \cup \{x_1u_1, \cdots, x_1u_t\} \). Then we can extend \( f \) to be a SDF \( g \) of \( T'+S' \) the same as the case \( f(v) = 1 \) and \( f(u) = -1 \) and so the result is valid.

If \( f(u) = -1 \), then \( f(u_1) + \cdots + f(u_t) \geq 3 \). Since \( t \leq r \leq 3 \), \( t = 3 \) and \( f(u_1) = f(u_2) = f(u_3) = 1 \). Let \( S' = (S - \{vu_1\}) \cup \{x_1u_1\} \) and define

\[
g(x) = \begin{cases} 
-1, & x = x_2 \\
1, & x = x_1, x_3 \\
f(x), & x \in V(T)
\end{cases}
\]

Then \( g \) is a SDF of \( T'+S' \) and \( \gamma_s(T'+S') \leq g(V(T'+S')) \leq \gamma_s(T+S) + 1 \leq \gamma_s(T) - 1 = \gamma_s(T') - 2 \). This also implies that \( R_s(T') \leq |S'| = |S| = R_s(T) \).

**Theorem 5.5.** For any tree \( T \) of order \( n \geq 2 \), \( R_s(T) \leq 2 \).
\textit{Proof.} We prove the result by induction on the order of \( T \). Since the result is true for \( T = K_2 \), we assume that \( n \geq 3 \). If \( n = 3 \), then, by Theorem 3.2, \( R_s(T) = 1 \) and the result is true. Now assume that \( n \geq 4 \) and the result is true for any tree with order less than \( n \).

Let \( T \) be a tree with \( |V(T)| = n \) and let \( f : V(T) \to \{-1,1\} \) be a minimum SDF of \( T \). Then \( f(V(T)) = \gamma_s(T) \) and \( f(v) = 1 \) for any \( v \in L(T) \cup S(T) \) by Lemma 2.2.

Let \( P_m = v_1v_2\cdots v_m \) be a longest path of \( T \).

If \( d(v_2) \geq 3 \), then there are at least two leaves adjacent with \( v_2 \) since \( P_m \) is a longest path of \( T \). Since \( f(v_2) = 1 \), \( f[v_2] \geq 3 - 1 = 2 \). By Lemma 5.3, \( R_s(T) \leq 2 \) and so the result is true. Hence, in the following, we suppose \( d(v_2) = 2 \).

\textbf{Case 1.} \( d(v_3) \geq 3 \).

\textbf{Case 1.1.} If \( v_3 \) is adjacent with a leaf \( x \), then \( f(x) = f(v_3) = 1 \). So \( f[v_2] \geq 3 \). By Lemma 5.2, \( R_s(T) = 1 \).

\textbf{Case 1.2.} If \( v_3 \) is not adjacent with any leaf of \( T \), since \( P_m \) is a longest path of \( T \), each neighbor of \( v_3 \) other than \( v_4 \) is a support vertex of \( T \). Since \( d(v_2) = 2 \), we can assume that each component of \( T - \{v_3\} \) not containing \( v_4 \) is isomorphic to \( K_2 \). If \( f(v_3) = 1 \), then \( f[v_2] \geq 3 \). By Lemma 5.2, \( R_s(T) = 1 \). Now we assume \( f(v_3) = -1 \).

Let \( y_1y_2 \) be a component of \( T - \{v_3\} \) other than \( v_1v_2 \) with \( y_2v_3 \in E(T) \). Let \( S = \{v_1v_3, y_1v_3\} \). Define a function \( g : V(T + S) \to \{-1,1\} \) as follows:

\[ g(x) = \begin{cases} 
-1, & x = y_1, v_1 \\
1, & x = v_3 \\
f(x), & \text{otherwise}
\end{cases} \]

It is an easy task to check that \( g[x] \geq 1 \) for any vertex \( x \in V(T + S) \) and hence \( g \) is a SDF of \( T + S \). So \( \gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2 \) which implies that \( R_s(T) \leq |S| = 2 \).

\textbf{Case 2.} \( d(v_3) = 2 \).

\textbf{Case 2.1.} If \( f(v_3) = 1 \), then \( f[v_2] \geq 3 \) and so \( R_s(T) = 1 \) by Lemma 5.2.

\textbf{Case 2.2.} If \( f(v_3) = -1 \), then, to guarantee \( f[v_3] \geq 1 \), \( f(v_4) \) must be 1.

If \( d(v_4) = 2 \), then, to guarantee \( f[v_4] \geq 1 \), \( f(v_5) = 1 \). Let \( T' = T - \{v_1, v_2, v_3\} \). By the inductive hypothesis, \( R_s(T') \leq 2 \). Since \( \{v_1, v_2, v_3\} \) induce a path \( P_3 \), by Lemma 5.4, \( R_s(T) = R_s(T' + P_3) \leq R_s(T') \leq 2 \).

Now assume that \( d(v_4) \geq 3 \).

If \( v_4 \) is a support vertex and \( w \) is a leaf adjacent with \( v_4 \), then \( f(w) = f(v_4) = 1 \). Let \( S = \{v_1v_3, vwv_3\} \). We can define a SDF \( g \) of \( T + S \) as follows:

\[ g(x) = \begin{cases} 
-1, & x = w, v_1 \\
1, & x = v_3 \\
f(x), & \text{otherwise}
\end{cases} \]

So \( \gamma_s(T + S) \leq g(V(T + S)) = \gamma_s(T) - 2 \) implying that \( R_s(T) \leq 2 \).
If \( v_4 \) is adjacent with a support vertex \( y \) such that \( N(y) - \{ v_4 \} \) are leaves of \( T \), then \( f(y) = 1 \). Since \( f(v_4) = 1 \) and the value of any leaf assigned by \( f \) is 1, \( f[y] \geq 3 \). By Lemma 5.2, we have \( R_s(T) = 1 \).

By the above proofs, we can assume that: (i) each component of \( T - \{ v_4 \} \) not containing \( v_5 \) is isomorphic to \( P_3 \) with an end adjacent with \( v_4 \); (ii) the value of the vertex adjacent with \( v_4 \) assigned by \( f \) is \(-1\). By this assumption, to guarantee \( f[v_4] \geq 1 \), there is exactly one such component, that means \( d(v_4) = 2 \), contradicts with the assumption \( d(v_4) \geq 3 \).

**Remark 5.6.** The upper bound \( R_s(T) \leq 2 \) is sharp since \( R(P_{3k+2}) = 2 \), \( k \geq 1 \).

**References**


